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GENERALIZED SUFFICIENT STRENGTH CRITERIA. DESCRIPTION OF THE PRE-FRACTURE ZONE

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A generalized sufficient discrete–integral strength criterion is proposed for opening mode cracks in structured media. The criterion is constructed for materials of two types: an elastoplastic material with restricted deformability and an elastic ideal-plastic material. A detailed description of the pre-fracture zone is given. The cross-sectional dimension of the plastic zone near the tip of the initial crack is used as the cross-sectional dimension of the pre-fracture zone. It is assumed that the critical opening of the initial crack depends on the deformability of plastic materials. Based on the ratio of the length of the pre-fracture zone to the length of the initial crack, the following fracture types are distinguished: brittle, quasibrittle, quasiviscous, and viscous. For the first three types, the fracture curves are described in detail. Exact and approximate equations are proposed that relate critical parameters to the theoretical strength of a granular material, grain size, and parameters characterizing the averaging interval and the damage of the initial and deformed material.

Introduction. Panasyuk et al. [1] studied comprehensively the dependence of the critical stress intensity factor (SIF) $K_{\rm Ic}$ on the standard mechanical characteristics of a material with allowance for the structure of this material. Results of this study are discussed in [2]. Necessary information on fracture criteria taking into account the structural parameters of materials can be found in [2, 3].

Following the Neuber–Novozhilov approach [4, 5], Kornev [6, 7] used the necessary fracture criterion [5] to determine the critical load beyond which a pre-fracture zone begins to form ahead of the crack tip in materials with structure. For the critical load predicted by the sufficient fracture criterion for a macrocrack [5], the length of the pre-fracture zone reaches the critical value and the solid is split into fragments (see [8–10]). It should be noted that the length of the pre-fracture zone (see [8–10]) in the Leonov–Panasyuk–Dugdale model [11, 12] is determined by standard mechanical characteristics.

1. Interrelation between the Criteria and Physicomechanical Models of the Pre-Fracture Zone. We consider a solid body with a structural hierarchy $i = 1, 2, \ldots, i_0$ (i_0 is the total number of structures) [6, 7]. Let the solid with a sharp crack be loaded in such a manner that the first mode of fracture under plane stresses occurs. An opening mode crack is modeled by a bilateral cut. It is assumed that of the critical stresses $\sigma_\infty^{0(i)}$ predicted by the necessary criterion, the minimum critical stress corresponds to the macrostructure with number $i = 1$, i.e.,

$$\min \sigma_\infty^{0(i)} = \sigma_\infty^{0(1)}, \tag{1}$$

and the critical stresses $\sigma_\infty^{*(1)}$ predicted by the sufficient criterion for a macrostructure satisfy the inequalities

$$\sigma_\infty^{*(1)} > \sigma_\infty^{0(1)}, \qquad \sigma_\infty^{*(1)} < \sigma_\infty^{0(i)} \qquad (i = 2, 3, \ldots, i_0). \tag{2}$$

The first inequality in (2) is obvious (the critical stresses predicted by the sufficient criterion are higher than those predicted by the necessary criterion for a macrostructure); if the subsequent inequalities are satisfied, pre-fracture zones are not formed for meso- and microstructures ($i = 2, 3, \ldots, i_0$). Moreover, we assume that for the critical stresses $\sigma_\infty^{*(1)}$, satellite cracks do not appear ahead of the crack tip (see [6, 7]). Thus, a pre-fracture zone is formed

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only for the microstructure $i = 1$ (its length is denoted by Δ_1), and other pre-fracture zones which generate satellite cracks are not formed ahead of this zone.

The necessary discrete–integral criterion of brittle strength (pre-fracture zone is absent: $\Delta_1 = 0$ and $K^0_{\mathrm{I}\infty} > 0$) has the form [6, 7]

$$x \geqslant 0\colon \quad \frac{1}{k_1 r_1} \int\limits_0^{n_1 r_1} \sigma_y(x,0)\,dx = \sigma_{m1}, \qquad x > 0\colon \quad \sigma_y(x,0) \simeq \sigma_\infty + \frac{K^0_{\mathrm{I}\infty}}{(2\pi x)^{1/2}}. \tag{3}$$

For a macrostructure, the sufficient discrete–integral criteria of quasibrittle, quasiviscous or viscous strength, which describe development of the pre-fracture zone, are written as follows:

— For the pre-fracture zone formed [8–10] (critical length of the pre-fracture zone $\Delta_1^* > 0, h_{m1} > 0, K_{\mathrm{I}} > 0$, and an elastoplastic material with restricted deformability), we have

$$\begin{gathered} x \geqslant 0\colon \quad \frac{1}{k_1 r_1} \int\limits_0^{n_1 r_1} \sigma_y(x,0)\,dx = \sigma_{m1}, \qquad x > 0\colon \quad \sigma_y(x,0) \simeq \sigma_\infty + \frac{K_{\mathrm{I}}}{(2\pi x)^{1/2}}, \\ x \leqslant 0\colon \quad 2v^* = ((æ+1)/G)K_{\mathrm{I}}\sqrt{\Delta_1^*/(2\pi)} = h_{m1}; \end{gathered} \tag{4}$$

— For the pre-fracture zone of critical length Δ_1^{**} [classical case of an elastic ideal-plastic material (see, e.g., [13]) and $K_{\mathrm{I}} = 0$], we have

$$\begin{gathered} x \geqslant 0\colon \quad \frac{1}{k_1 r_1} \int\limits_0^{n_1 r_1} \sigma_y(x,0)\,dx = \sigma_{m1}, \qquad \sigma_y(x,0) \simeq \sigma_{m1} + \frac{K_{\mathrm{I}}}{(2\pi x)^{1/2}}, \\ x \leqslant 0\colon \quad \delta_{m1} = \pi l_0(\sigma_\infty/E)(\sigma_\infty/\sigma_{m1}); \end{gathered} \tag{5}$$

$$K_{\mathrm{I}} = K_{\mathrm{I}\infty} + K_{\mathrm{I}\Delta_1}, \quad K_{\mathrm{I}\infty} > 0, \quad K_{\mathrm{I}\Delta_1} < 0, \quad K^0_{\mathrm{I}\infty} = K^0_{\mathrm{I}\infty}(l_0), \quad K_{\mathrm{I}\infty} = K_{\mathrm{I}\infty}(l). \tag{6}$$

In relations (3)–(6), σ_y are normal stresses on the continuation of the crack [they have a singular component for criteria (3) and (4)], Oxy is a Cartesian coordinate system whose origin lies at the right tip of the sharp macrocrack, r_1 is the characteristic linear dimension of the material's macrostructure, for example, the grain size of a polycrystalline material, n_1 and k_1 are numbers ($n_1 \geqslant k_1$, where $k_1 \geqslant 1$ is the number of defect-free grains), $n_1 r_1$ is the averaging interval for the granular material ($n_1 \geqslant 1$), $(n_1 - k_1)/n_1$ is the coefficient that takes into account the damage of the granular material in the interval $n_1 r_1$, σ_{m1} is the "theoretical" strength of the granular material (yield point), $h_{m1} = 2v^*(-\Delta_1)$ is the critical opening of the crack for $\Delta_1 = \Delta_1^*$, $2v(x)$ is the crack opening, $æ = (3 - \nu)/(1 + \nu)$ for plane stresses, ν is the Poisson's ratio, E and G are the Young's and shear moduli, respectively, δ_{m1} is the critical crack opening for $\Delta_1 = \Delta_1^{**}$, l_0 and $l = l_0 + \Delta_1 l = l_0 + \Delta_1$ are the halflengths of the initial and model internal cracks, respectively [the halflength of the initial crack is used in criterion (3) and the halflength of the model crack l is used in criteria (4) and (5)], the total SIF $K_{\mathrm{I}} \geqslant 0$ is calculated from relation (6) for the corresponding problem, $K_{\mathrm{I}\infty}$ is the SIF generated by the remote stresses σ_∞ ($K_{\mathrm{I}\infty} = \sigma_\infty\sqrt{\pi l}$ for an internal crack), and $K_{\mathrm{I}\Delta_1}$ is the SIF generated by the stresses $\eta_1\sigma_{m1}$ in the vicinity of the crack tip in the pre-fracture zone in accordance with the Leonov–Panasyuk–Dugdale model [11, 12]. The parameter η_1 characterizing the damage of the material in the pre-fracture zone (see, e.g., [14, Fig. 4.2.4] and [15, Fig. 35]) can be related to the plastic loosening of the material. Four main types of fracture are shown in [16, Fig. 1]. Hereafter, subscript 1 is omitted (except for the parameter r_1, which characterizes the linear dimension of the material macrostructure) since a pre-fracture zone is formed only in a macrostructure [see relations (1) and (2)].

Remark 1. To formulate criteria (3) and (4), we use the relation $\sigma_y(x,0) \simeq \sigma_\infty + K^0_{\mathrm{I}\infty}(2\pi x)^{-1/2}$, where $x > 0$. This function coincides with the exact solution $\sigma_y(x,0) = \sigma_\infty(x + l_0)[x(x + 2l_0)]^{-1/2}$ ($x > 0$) for an internal crack as $x \to \infty$ and has the same singularity for $x \to 0$.

The criteria (3)–(5) proposed above can be classified by the number of parameters used [3]: criteria (3) and (5) are one-parameter force criteria (the parameter σ_m is often related to the yield point) and criterion (4) is a two-parameter strain-force criterion (these parameters are σ_m and h_m). Thus, criterion (4) contains not only the yield point but also the parameter h_m, which characterizes the critical crack opening and depends on the deformability of the material. It is obvious that in the limiting passage from an elastoplastic material with restricted deformability [criterion (4)] to an elastic ideal-plastic material [criterion (5)], part of useful information is lost. It is therefore expedient to combine criteria (4) and (5) to obtain a generalized sufficient criterion.

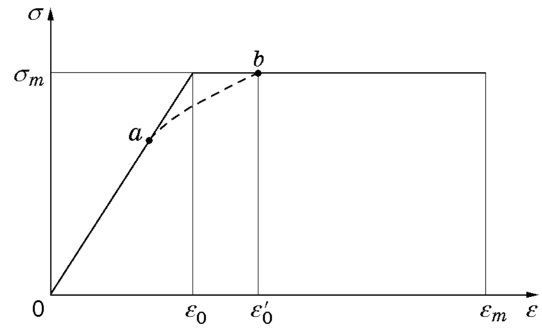

Fig. 1

2. Types of Fracture. Criterion (3) describes brittle fracture (first type), whereas criteria (4) and (5) describe quasibrittle fracture (second type) and are applicable for quasiviscous and viscous fracture (third and fourth types, respectively) depending on which type of fracture is realized (see [16]). The second, third, and fourth types of fracture do not contradict the Leonov–Panasyuk–Dugdale model [11, 12] or its modification. We describe quantitatively the four main types of fracture determined by the relative length of the pre-fracture zone Δ/l_0: 1) $\Delta \equiv 0$; 2) $\Delta/l_0 = o(1)$; 3) $\Delta/l_0 = O(1)$ or $l_0/\Delta = O(1)$; 4) $l_0/\Delta = o(1)$. For brittle fracture, a pre-fracture zone is absent; for quasibrittle fracture, the length of the pre-fracture zone is much smaller than the length of the initial crack: $\Delta \ll l_0$; for quasiviscous fracture, the length of the pre-fracture zone is comparable with the length of the initial crack: $\Delta \approx l_0$; for viscous fracture, the length of the initial crack is much smaller than the length of the pre-fracture zone: $l_0 \ll \Delta$.

The above classification corresponds to the Leonov–Panasyuk–Dugdale model [11, 12] if the plastic zones near the tips of an internal crack do not merge and the boundary of the plastic zone of an edge crack does not reach the surface of the solid. These restrictions can be of fundamental importance for the quasiviscous and viscous types of fracture.

Figure 1 shows σ–ε diagrams for elastic ideal-plastic and nonlinear-elastoplastic materials. The segment of nonlinear deformation ab is shown by a dashed curve (σ_m is the yield point, ε_0 and ε_0' are the limiting relative elongations of the elastic and nonlinear-elastic materials, respectively, and ε_m is the limiting elongation of the material; for the elastic ideal-plastic material, it is assumed that $\varepsilon_m = \infty$). Of fundamental importance for further consideration are the parameters $\varepsilon_m - \varepsilon_0$ and $\varepsilon_m - \varepsilon_0'$, which characterize the maximum relative elongations of the plastic material and determine the critical crack opening if the cross-sectional dimension of the pre-fracture zone h is known. The proposed model of a material ignores differences between the σ–ε diagrams of the elastic ideal-plastic and nonlinear elastoplastic materials.

Remark 2. It should be kept in mind that for a nonlinear-elastoplastic material, the stress-field singularities at the crack tip can change [17].

3. Critical Crack Opening. We determine the critical crack opening (CCO) for materials of two types. For an elastoplastic material with restricted deformability, the critical parameters have the form $\Delta = \Delta^*$ and $\sigma_\infty = \sigma_\infty^*$, and for an elastic ideal-plastic material, the critical parameters are $\Delta = \Delta^{**}$ and $\sigma_\infty = \sigma_\infty^{**}$. The proposed critical parameters Δ^* or Δ^{**} cannot be compared directly with the critical parameter of the CCO criterion (see [3, Sec. 3]) since the linear dimension that characterizes the cross section of the pre-fracture zone is unknown. We estimate the cross-sectional dimension of the pre-fracture zone h in the vicinity of the crack tip. As the dimension h we use the cross-sectional dimension of the plastic zone near the tip of an initial crack of length l_0. It should be noted that large plastic strains are typical of the failure of plastic metals [15].

3.1. *Quasibrittle Fracture* ($\Delta^* \ll l_0$ and $\sigma_\infty^* \ll \sigma_m$ or $\Delta^{**} \ll l_0$ and $\sigma_\infty^{**} \ll \sigma_m$). We assume that the remote stresses are much lower than the yield point: $\sigma_\infty^* \ll \sigma_m$ or $\sigma_\infty^{**} \ll \sigma_m$. In this case, we can use Irwin's plastic-strain correction. The estimate of the plastic zone size coincides with that for the plane stresses (see, e.g., [13]):

$$\rho(\theta) = (K_{\mathrm{I}\infty}^0)^2((3/2)\sin^2\theta + 1 + \cos\theta)/(4\pi\sigma_m^2), \qquad K_{\mathrm{I}\infty}^0 = K_{\mathrm{I}\infty}^0(l_0), \tag{7}$$

where ρ is the radius vector and θ is the polar angle. Since $\theta = \pi/2$ at the crack tip, from (7) we obtain the cross-sectional dimension of the pre-fracture zone for plane stresses

$$h = 2\rho(\pi/2) = 5(K_{\mathrm{I}\infty}^0)^2/(4\pi\sigma_m^2). \tag{8}$$

For elastoplastic and nonlinear elastoplastic materials, the critical crack opening $2v^*(-\Delta) = h_m$ is determined, respectively, by the formulas

$$h_m = 5(K^0_{\mathrm{I}\infty})^2(\varepsilon_m - \varepsilon_0)/(4\pi\sigma_m^2), \qquad h_m = 5(K^0_{\mathrm{I}\infty})^2(\varepsilon_m - \varepsilon_0')/(4\pi\sigma_m^2). \tag{9}$$

The parameters of the CCO h_m in relation (9) are related to the standard characteristics of the σ–ε diagrams of the materials (Fig. 1). For internal cracks, the CCO in the above-mentioned materials is given by formulas

$$h_m = 5(\sigma^*_\infty/\sigma_m)^2 l_0(\varepsilon_m - \varepsilon_0)/4, \qquad h_m = 5(\sigma^*_\infty/\sigma_m)^2 l_0(\varepsilon_m - \varepsilon_0')/4. \tag{9$'$}$$

For an elastoplastic material with restricted deformability, the inequality $\varepsilon_m < \infty$ holds in the presence of plasticity, and for an elastic ideal-plastic material, we have $\varepsilon_m = \infty$. The following relation is valid: $\Delta^* \leqslant \Delta^{**}$. Whether or not the pre-fracture zone reaches the critical length Δ^{**} depends on the plasticity margin of the material or, more precisely, its deformability margin. For an elastoplastic material with restricted deformability for plasticity with $K^*_{\mathrm{I}} > 0$, the length of the pre-fracture zone Δ cannot exceed the critical length Δ^*. For an elastoplastic material with restricted deformability for plasticity with $K^{**}_{\mathrm{I}} = 0$, the length of the pre-fracture zone Δ coincides with Δ^{**}. For an elastic ideal-plastic material for $K_{\mathrm{I}} = 0$, we always have Δ^{**} since $\varepsilon_m = \infty$.

3.2. *Quasiviscous and Viscous Types of Fracture* [$\Delta^* \approx l_0$, $\sigma^*_\infty = O(\sigma_m)$, $\Delta^{**} \approx l_0$, $\sigma^{**}_\infty = O(\sigma_m)$, $l_0 < \Delta^*$, $\sigma^*_\infty \approx \sigma_m$, $l_0 < \Delta^{**}$, and $\sigma^{**}_\infty \approx \sigma_m$ but $\sigma^*_\infty < \sigma_m$ and $\sigma^{**}_\infty < \sigma_m$]. Relations (7) and (8) are not applicable to viscous fracture since they are obtained under the restriction $\sigma^*_\infty \ll \sigma_m$ or $\sigma^{**}_\infty \ll \sigma_m$; relations (7) and (8) can be used in the case of quasiviscous fracture for $\sigma^*_\infty < \sigma_m$ or $\sigma^{**}_\infty < \sigma_m$.

4. Elastic Ideal-Plastic Material ($\boldsymbol{\varepsilon_m = \infty}$). We study the behavior of an elastic ideal-plastic material in the vicinity of a crack tip. Using the condition that the total SIF of an internal crack K^{**}_{I} vanishes, we obtain the following equation for the critical length of the pre-fracture zone Δ^{**} (the total SIF cannot be negative since the crack edges overlap for $K_{\mathrm{I}} < 0$, which can easily be verified [18])

$$K^{**}_{\mathrm{I}} = K^{**}_{\mathrm{I}\infty} + K^{**}_{\mathrm{I}\Delta} = 0, \qquad K^{**}_{\mathrm{I}\infty} = \sigma^{**}_\infty\sqrt{\pi l^{**}}, \qquad l^{**} = l_0 + \Delta^{**},$$
$$K^{**}_{\mathrm{I}\Delta} = -(2\sigma_m\sqrt{l^{**}}/\sqrt{\pi})\arccos(1 - \Delta^{**}/l^{**}) = -\sigma_m\sqrt{\pi l^{**}}\,[1 - (2/\pi)\arcsin(1 - \Delta^{**}/l^{**})]. \tag{10}$$

The critical crack opening at the tip of the initial crack δ_m can be written in the form [13]

$$\delta_m = (8\sigma_m l_0/(\pi E))\ln(\sec(\pi\sigma^{**}_\infty/(2\sigma_m))), \qquad n = k = 1, \quad \eta_1 = 1. \tag{11}$$

Remark 3. The classical sufficient strength criterion (5) is reformulated so as to describe fracture of structured bodies. However, formal calculations coincide with classical results if $n = k = 1$ and $\eta = 1$.

This case is considered below.

Obviously, the critical length of the pre-fracture zone Δ^{**} ahead of the initial crack l_0 (or in the case of a fictitious crack l^{**}) can be written in the following form [see (10)]:

$$\Delta^{**}/l_0 = \sec(\pi\sigma^*_\infty/(2\sigma_m)) - 1, \qquad \Delta^{**}/l^{**} = 1 - \cos(\pi\sigma^*_\infty/(2\sigma_m)). \tag{12}$$

4.1. *Quasibrittle Fracture* ($\Delta^{**} \ll l_0$ and $\sigma^*_\infty \ll \sigma_m$). In the quasibrittle approximation, after some transformations in relations (12), we obtain the critical length of the pre-fracture zone Δ^{**} for the initial (l_0) or fictitious (l^{**}) cracks by retaining the leading terms of the expansions ($\sigma_\infty/\sigma_m \ll 1$):

$$\Delta^{**}/l_0 = (\pi^2/8)(\sigma^{**}_\infty/\sigma_m)^2, \qquad \Delta^{**}/l^{**} = (\pi^2/8)(\sigma^{**}_\infty/\sigma_m)^2. \tag{13}$$

The opening at the initial-crack tip δ_m is obtained by retaining the first term in series (11) (see [13]):

$$\delta_m = \pi l_0(\sigma^{**}_\infty/E)(\sigma^{**}_\infty/\sigma_m). \tag{14}$$

It should be noted that the extent of the pre-fracture zone Δ^{**} in (13) is approximately equal to the extent of the plastic zone, and formula (14) for the opening at the initial-crack tip δ_m is frequently used in experimental fracture mechanics [13].

Let us compare the quantities δ_m and h_m, which characterize the initial-crack opening in the sufficient criteria (5) and (4). For $K^*_{\mathrm{I}} \to 0$, we obtain $\delta_m \approx h_m$; using the first relation in (9$'$) and relation (14), we finally obtain the approximate equality

$$\varepsilon_m - \varepsilon_0 \approx 4\pi\sigma_m/(5E), \tag{15}$$

which separates the regions of applicability of these criteria for internal cracks. Figure 2 shows the regions of applicability of the sufficient criteria (4) and (5) for internal cracks in accordance with (15). These regions are

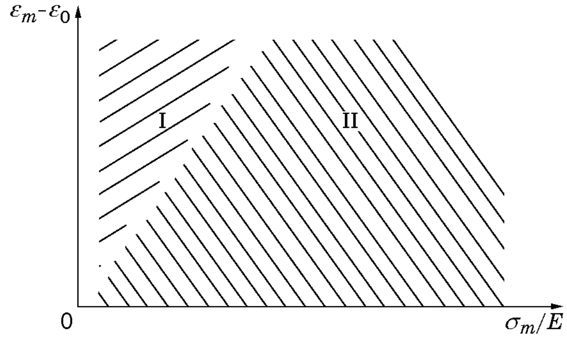

Fig. 2

determined by the relation between the "theoretical" strength, modulus of elasticity, and deformability of the plastic material. In Fig. 2, region I refers to the classical criterion of quasibrittle strength (5), region II to the criterion of quasibrittle strength (4), and the axis $\varepsilon_m - \varepsilon_0 = 0$ to the criterion of brittle strength (3). Only few real materials correspond to region I (for example, gold and some plastics), most metal structural materials correspond to region II, and almost all ceramic materials correspond to the part of region II adjacent to the axis $\varepsilon_m - \varepsilon_0 = 0$.

4.2. *Quasiviscous Fracture* [$\Delta^{**} \approx l_0$ and $\sigma_\infty^{**} = O(\sigma_m)$ but $\sigma_\infty^{**} < \sigma_m$]. We obtain an analytical expression for the critical length of the pre-fracture zone Δ^{**} and the opening at the tip of the initial crack δ_m. Retaining two terms in expansions for (11) and (12), we obtain

$$\frac{\Delta^{**}}{l_0} = \frac{\pi^2}{8}\Big(\frac{\sigma_\infty^{**}}{\sigma_m}\Big)^2\Big[1 + \frac{5\pi^2}{48}\Big(\frac{\sigma_\infty^{**}}{\sigma_m}\Big)^2\Big], \qquad \frac{\Delta^{**}}{l^{**}} = \frac{\pi^2}{8}\Big(\frac{\sigma_\infty^{**}}{\sigma_m}\Big)^2\Big[1 - \frac{\pi^2}{48}\Big(\frac{\sigma_\infty^{**}}{\sigma_m}\Big)^2\Big],$$
$$\delta_m = \pi l_0(\sigma_\infty^{**}/E)(\sigma_\infty^{**}/\sigma_m)[1 + (\pi^2/24)(\sigma_\infty^{**}/\sigma_m)^2]. \tag{16}$$

Obviously, the second terms in square brackets in (16) are corrections to the main part of the solution. As in Sec. 3, we compare the quantities δ_m and h_m. Then, for internal cracks, we obtain

$$\varepsilon_m - \varepsilon_0 \approx (4\pi/5)(\sigma_m/E)[1 + (\pi^2/24)(\sigma_\infty^{**}/\sigma_m)^2]. \tag{17}$$

The last term on the right side of relation (17) depends on the ratio $\sigma_\infty^{**}/\sigma_m$. For quasiviscous fracture, the regions of applicability of the sufficient criteria (4) and (5) [see relations (15) and (17) and Fig. 2] are shifted slightly compared to such regions for quasibrittle fracture.

5. Elastoplastic Material with Restricted Deformability ($\varepsilon_m < \infty$). If the deformability of an elastoplastic material is restricted, the equation for the critical length of the pre-fracture zone Δ^* follows from the last relation of the sufficient criterion (4), the total SIF $K_{\rm I}$ of the internal crack being nonzero ($K_{\rm I} > 0$) [18]:

$$K_{\rm I} = K_{{\rm I}\infty} + K_{{\rm I}\Delta}, \qquad K_{{\rm I}\infty} = \sigma_\infty\sqrt{\pi l}, \qquad l = l_0 + \Delta,$$
$$K_{{\rm I}\Delta} = -(2\eta\sigma_m\sqrt{l}/\sqrt{\pi})\arccos(1 - \Delta/l) = -\eta\sigma_m\sqrt{\pi l}[1 - (2/\pi)\arcsin(1 - \Delta/l)]. \tag{18}$$

It is obvious [see (18)] that the total and critical total SIFs $K_{\rm I}$ and $K_{\rm I}^*$ are complex functions of the length Δ and the critical length of the pre-fracture zone and Δ^*, respectively: $K_{\rm I} = K_{\rm I}(\Delta)$ and $K_{\rm I}^* = K_{\rm I}^*(\Delta^*)$.

After approximate transformation of criterion (4), we obtain two nonlinear equations that relate the critical parameters $K_{\rm I}^*$, Δ^*, σ_∞^*, and h_m:

$$\frac{K_{\rm I}^*}{\sigma_\infty^*\sqrt{r_1}} = \sqrt{\frac{\pi}{2}n}\Big(\frac{\sigma_m}{\sigma_\infty^*}\frac{k}{n} - 1\Big), \qquad \Delta^* = 2\pi\Big(\frac{G}{æ+1}\frac{h_m}{K_{\rm I}^*}\Big)^2. \tag{19}$$

Finally, we obtain the equation for the critical length of the pre-fracture zone Δ^* by substituting expression (18) for the critical total SIF $K_{\rm I}^*$ into the second equation (19):

$$\Big\{\frac{\sigma_\infty^*}{\sigma_m} - \eta\Big[1 - \frac{2}{\pi}\arcsin\Big(1 - \frac{\Delta^*}{l^*}\Big)\Big]\Big\}\sqrt{\frac{\Delta^*}{l^*}} = \frac{\sqrt{2}}{æ+1}\frac{G}{\sigma_m}\frac{h_m}{l^*}. \tag{20}$$

The critical length of the pre-fracture zone Δ^* satisfies the natural restrictions $0 < \Delta^*/l^* < 1$ for $l_0 > 0$ and $l^* = l_0 + \Delta^*$. No other restrictions were imposed on the critical length of the pre-fracture zone Δ^* in relations (19) and (20). It is worth noting that relations (19) and (20) contain only standard stiffness and strength characteristics and also the linear dimension r_1 that characterizes the grain size of the macrostructure and the damage parameter η that takes into account the plastic loosening of the material.

5.1. *Quasibrittle Fracture* ($\Delta^* \ll l_0$ and $\sigma_\infty^* \ll \sigma_m$). We determine the critical length of the pre-fracture zone Δ^* for quasibrittle fracture. If $\Delta^* \ll l_0$, then $\Delta^* \ll l^*$. Hence, the last relation in (20) can be simplified substantially taking into account that $\arcsin(1 - \Delta^*/l^*) \simeq \pi/2 - \sqrt{2\Delta^*/l^*}$. Finally, we obtain the following quadratic equation for the parameter $\sqrt{\Delta^*/l^*}$:

$$\Big(\sqrt{\frac{\Delta^*}{l^*}}\Big)^2 - \frac{\pi}{2\sqrt{2}} \frac{\sigma_\infty^*}{\eta\sigma_m} \sqrt{\frac{\Delta^*}{l^*}} + \frac{\pi}{2(æ+1)} \frac{G}{\eta\sigma_m} \frac{h_m}{l^*} \simeq 0.$$

Neglecting quantities of high order of smallness compared to unity, we express the smaller root of the quadratic equation in explicit form

$$\sqrt{\frac{\Delta^*}{l^*}} \simeq \frac{\sqrt{2}}{æ+1} \frac{G}{\sigma_\infty^*} \frac{h_m}{l^*}.$$

The last relation implies that the relative dimension of the pre-fracture zone $\sqrt{\Delta^*/l^*}$ depends linearly on the critical crack opening h_m.

For convenience, the approximate algebraic equations relating the critical parameters σ_∞^*, l^*, Δ^*, and h_m to the theoretical strength σ_m and grain size r_1 of a granular material, and the parameters n, k, and η, which characterize the averaging interval and the damage of the initial and plastically deformed material, respectively, can be written in one of the following forms:

$$\frac{2l^*}{r_1} \simeq \Big(\frac{\sigma_m}{\sigma_\infty^*} - \frac{n}{k}\Big)^2 \frac{k^2}{n} \Big(1 - \frac{2\sqrt{2}}{\pi} \eta \frac{\sigma_m}{\sigma_\infty^*} \sqrt{\frac{\Delta^*}{l^*}}\Big)^{-2}, \quad \sqrt{\frac{\Delta^*}{l^*}} \simeq \frac{\sqrt{2}}{æ+1} \frac{G}{\sigma_\infty^*} \frac{h_m}{l^*},$$
$$\frac{\sigma_\infty^*}{\sigma_m} \simeq \Big[\frac{\sqrt{n}}{k} \sqrt{\frac{2l^*}{r_1}} \Big(1 - \frac{2\sqrt{2}}{\pi} \eta \frac{\sigma_m}{\sigma_\infty^*} \sqrt{\frac{\Delta^*}{l^*}}\Big) + \frac{n}{k}\Big]^{-1}. \tag{21}$$

The fracture curves for an elastoplastic material with restricted deformability (21) become the fracture curves for a brittle material if the length of the pre-fracture zone tends to zero: $\Delta^* \to 0$.

5.2. *Quasiviscous Fracture* [$\Delta^* \approx l_0$ and $\sigma_\infty^* = O(\sigma_m)$ but $\sigma_\infty^* < \sigma_m$]. The simplifications in Sec. 5.1 cannot be used to study the pre-fracture zone for quasiviscous fracture. Therefore, one has to solve Eq. (20) numerically under the natural restrictions $0 < \Delta^*/l^* < 1$ for specified initial parameters of this equation. The root Δ^*/l^* is used to find the critical SIF K_{I}^*. The transcendental equation relating the critical parameters σ_∞^*, l^*, Δ^*, and h_m to the theoretical strength of a granular material σ_m, the grain size r_1, and the parameters n, k, and η has the following form [see the first equation in (19)]:

$$\sqrt{\frac{2l^*}{r_1}} \Big\{1 - \sqrt{2}\eta \frac{\sigma_m}{\sigma_\infty^*} \Big[1 - \frac{2}{\pi} \arcsin\Big(1 - \frac{\Delta^*}{l^*}\Big)\Big]\Big\} = \frac{k}{\sqrt{n}} \Big(\frac{\sigma_m}{\sigma_\infty^*} - \frac{n}{k}\Big). \tag{22}$$

The structure of the transcendental equation (22) is similar to that of Eqs. (21); therefore, it is expedient to choose the root of Eqs. (21) as a zero approximation to find the roots of Eq. (22) by the method of successive approximations.

6. Discussion. Fracture curves were constructed for elastoplastic materials with unrestricted deformability within the framework of the Leonov–Panasyuk–Dugdale model [11, 12]. To construct these curves, we used information on a crack with connections between the edges in the vicinity of the crack tip for isotropic materials (see also [19]). The generalized sufficient criterion (4)–(6) fills the gap between the description of fracture of brittle bodies with structure [see criterion (3)] and that of elastoplastic bodies with restricted deformability. The sufficient criterion (4) for a structured elastoplastic material with restricted deformability is a two-parameter criterion. Equations (21) and (22) contain two parameters of the material (in addition to the structural parameter r_1): the theoretical strength (yield point) σ_m and the critical crack opening h_m. The approach proposed does not contradict Griffith's idea: for brittle structured bodies, the necessary criterion (3) yields the SIF $K_{\mathrm{I}}^*|_{\Delta^*=0}$ [see the first relation in (21)].

The critical crack opening h_m is expressed in terms of the limiting relative elongation of the material ε_m. In the limiting case $\varepsilon_m \to \varepsilon_0$, we have a brittle material, and in the case $\varepsilon_m \to \infty$, we have an elastic ideal-plastic material (classical solution).

The sufficient criterion (4)–(6) contains the material-structure parameter r_1, the theoretical strength (yield point) of the structured material σ_m, and the limiting relative elongation of the material ε_m. The material-structure parameter r_1 is determined by standard methods of physics of metals. It is expedient to determine the theoretical tensile strength σ_m for three-point bending of specimens whose cross-sectional dimension ranges from $20r_1$ to $50r_1$. In this case, the specimen surface should be processed appropriately to reduce the scatter of experimental data. The limiting relative elongation of the material ε_m is obtained by constructing a standard σ–ε diagram.

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